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# Engineering Notes

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## State-Space System Identification from Closed-Loop Frequency Response Data

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### Introduction

EXTENSIVE research has been conducted in active control of dynamic systems. Achieving high control performance on these systems usually requires an accurate model. Such a model can be derived from system identification techniques using experimental data. Recently, a method was developed to identify a state-space model from frequency response data for open-loop systems by using the state-space frequency domain identification algorithm.[1] This method uses a matrix-fraction for the curve fitting, and the curve fitting is reformulated as a linear problem that can be solved by the ordinary least-squares method in one step.

A different method has been proposed to identify a state-space plant from closed-loop I/O time-domain data with known feedback dynamics.[2] This Note is an extension of this time-domain closed-loop identification method to frequency domain. The method can identify a linear open-loop stochastic system from closed-loop frequency response test data with known feedback dynamics. The relationship between the frequency response function (FRF) and the closed-loop system and Kalman-filter Markov parameters is derived for linear stochastic systems. Once the closed-loop system and Kalman-filter Markov parameters are obtained from FRF, a recursive formula for computing the open-loop system and the Kalman-filter Markov parameters from the closed-loop system, Kalman filter and controller Markov parameters can be used. Finally, the open-loop system can be realized from the calculated open-loop system Markov parameters.

### Linear State-Space and FRF Relationship

A finite-dimensional, linear, discrete-time, time-invariant system can be modeled as

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

$$y_k = Cx_k + v_k \tag{2}$$

where $x \in R^{n\times 1}$, $u \in R^{s\times 1}$, $y \in R^{m\times 1}$ are state, input, and output vectors, respectively; $w_k$ is the process noise; $v_k$ is the measurement noise and $[A, B, C]$ are the state-space parameters. Sequences $w_k$ and $v_k$ are assumed Gaussian, white, and stationary with zero mean and covariance matrices $Q$ and $R$, respectively. One can derive a steady-state filter innovation model[1]:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + AK\varepsilon_k \tag{3}$$

$$y_k = C\hat{x}_k + \varepsilon_k \tag{4}$$

where $\hat{x}_k$ is the a priori estimated state, $K$ is the steady-state Kalman-filter gain, and $\varepsilon_k$ is the residual after filtering: $\varepsilon_k = y_k - C\hat{x}_k$. On the other hand, a dynamic output feedback controller can be modeled as

$$p_{k+1} = A_d p_k + B_d y_k \tag{5}$$

$$u_k = C_d p_k + D_d y_k + r_k \tag{6}$$

where $A_d$, $B_d$, $C_d$, and $D_d$ are the system matrices of the controller; $p \in R^{l\times 1}$ is the controller state vector; and $r \in R^{s\times 1}$ is the reference input to the closed-loop system. Combining Eqs. (3) and (6), the augmented closed-loop system dynamics become

$$\eta_{k+1} = A_c\eta_k + B_c r_k + A_c K_c \varepsilon_k \tag{7}$$

$$y_k = C_c\eta_k + \varepsilon_k \tag{8}$$

or

$$\eta_{k+1} = \bar{A}\eta_k + B_c r_k + A_c K_c y_k \tag{9}$$

where

$$\eta_k = \begin{bmatrix} \hat{x}_k \\ p_k \end{bmatrix}, \qquad A_c = \begin{bmatrix} A + BD_dC & BC_d \\ B_dC & A_d \end{bmatrix}$$

$$B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad A_cK_c = \begin{bmatrix} AK + BD_d \\ B_d \end{bmatrix}$$

$$C_c = [\,C \quad 0\,], \qquad \text{and} \qquad \bar{A} = A_c - A_cK_cC_c$$

The $z$ transforms of Eqs. (8) and (9) yield

$$y(z) = C_c\eta(z) + \varepsilon(z) \tag{10}$$

$$\eta(z) = (zI_t - \bar{A})^{-1}[A_cK_cy(z) + B_cr(z)] \tag{11}$$

where $I_t$ is an identity matrix with dimension $t = n + l$. Substituting Eq. (11) into Eq. (10), one obtains

$$y(z) = \left[I_m - C_c(zI_t - \bar{A})^{-1}A_cK_c\right]^{-1}C_c(zI_t - \bar{A})^{-1}B_cr(z) + \left[I_m - C_c(zI_t - \bar{A})^{-1}A_cK_c\right]^{-1}\varepsilon(z) \tag{12}$$

The $z$ transforms of the dynamic output feedback controller (5) and (6) and the closed-loop state-space model (7) and (8) yield

$$u(z) = \sum_{k=0}^{\infty} Y_d(k)z^{-k}y(z) + r(z) \tag{13}$$

$$y(z) = \sum_{k=1}^{\infty} Y_c(k)z^{-k}r(z) + \sum_{k=0}^{\infty} N_c(k)z^{-k}\varepsilon(z) \tag{14}$$

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where $Y_d(k) = C_d A_d^{k-1} B_d$ are the controller Markov parameters, $Y_c(k) = C_c A_c^{k-1} B_c$ are the closed-loop-system Markov parameters, and $N_c(k) = C_c A_c^{k-1} A_c K_c$ are the closed-loop Kalman-filter Markov parameters. Note also that $Y_d(0) = D_d$ and $N_c(0) = I_m$.

The transfer-function matrix of the system described by Eqs. (12) and (14) is

$$G(z^{-1})\left[I_m - C_c(zI_t - \bar{A})^{-1}A_cK_c\right]^{-1} C_c(zI_t - \bar{A})^{-1}B_c = \sum_{k=1}^{\infty} Y_c(k)z^{-k} \quad (15)$$

The FRF is simply the transfer function matrix $G(z^{-1})$ calculated along the unit circle in the $z$ plane. It is also chosen that the transfer function matrix can be expressed by a left-fraction description[1] as

$$G(z^{-1}) = \alpha^{-1}(z^{-1})\beta(z^{-1}) \quad (16)$$

where both $\alpha(z^{-1})$ and $\beta(z^{-1})$ are matrix polynomials and can be found as a solution of the least-squares method. From Eq. (7.19) in Ref. 1, one has

$$\left(\sum_{i=0}^{p} \alpha_i z^{-i}\right)\left[\sum_{i=1}^{\infty} Y_c(i)z^{-i}\right] = \sum_{i=0}^{p} \beta_i z^{-i}$$

$$\alpha_0 = I_m, \qquad \beta_0 = 0 \quad (17)$$

From this relation, the closed-loop-system Markov parameters can be recursively calculated from the estimated $\alpha$ and $\beta$ matrix polynomials by using the parameter convolution of polynomial products as follows:

$$Y_c(k) = \beta_k - \sum_{i=1}^{k} \alpha_i Y_c(k-i) \quad (18)$$

Similarly, the closed-loop Kalman-filter Markov parameters can be recursively calculated from the estimated $\alpha$ matrix polynomials as follows:

$$N_c(k) = -\sum_{i=1}^{k} \alpha_i N_c(k-i) \quad (19)$$

Then, from the closed-loop-system Markov parameters $Y_c(k)$ and the closed-loop Kalman-filter Markov parameters $N_c(k)$, one can recursively calculate[2] the open-loop-system Markov parameters $Y(k) = CA^{k-1}B$ and the open-loop Kalman-filter Markov parameters $N(k) = CA^{k-1}AK$ with the known controller Markov parameters $Y_d(k) = C_d A_d^{k-1} B_d$,

$$Y(j) = Y_c(j) - \sum_{k=1}^{j}\sum_{i=1}^{k} Y(i)Y_d(k-i)Y_c(j-k) \quad (20)$$

$$N(j) = N_c(j) - \sum_{k=1}^{j}\sum_{i=1}^{k} Y(i)Y_d(k-i)N_c(j-k) \quad (21)$$

Note that $Y_d(0) = D_d$, $N_c(0) = I_m$, and $Y_c(0) = \beta_0$. The open-loop state-space model can be realized from the open-loop-system Markov parameters through the singular value decomposition method.[1] Once the open-loop $A$ and $C$ are obtained, one can easily calculate the open-loop Kalman filter gain from the open-loop Kalman-filter Markov parameters $N(k)$ in a least-squares sense.[2]

## Numerical and Test Example

An example is provided that consists of numerical simulations and actual hardware tests to validate the feasibility of the proposed frequency-domain closed-loop identification method. The large-gap magnetic suspension system[2] consists of a planar array of five copper electromagnets that actively suspend a small cylinder with a permanent magnet core. The cylinder is a rigid body and has six independent degrees of freedom, namely, three displacements ($x$, $y$, and $z$) and rotations (pitch, yaw, and roll). The roll of the cylinder is uncontrollable, and is assumed to be motionless. Because it is difficult to accurately model the magnetic field and its gradients, the analytical model needs to be improved through identification from experimental data. The discrete-time state-space parameters of the system and the dynamic output feedback using a sampling rate of 250 Hz are shown in the Appendix.

**Table 1 Comparison of eigenvalues of discrete-time analytical and identified model**

| Analytical model | Identified from simulation (1% noise variance) | Identified from testing |
|---|---|---|
| 1.1687 | 1.1686 | 1.2892 |
| 1.1629 | 1.1595 | 1.2796 |
| 1.0101 | 1.0019 | 1.0327 |
| 0.9810 | 0.9794 | 0.9042 |
| $0.9977 \pm 0.0257i$ | $0.9051 \pm 0.0983i$ | $1.0094 \pm 0.0341i$ |
| $0.9920 \pm 0.0133i$ | $0.8546 \pm 0.1947i$ | $0.9972 \pm 0.0221i$ |
| $0.8633 \pm 0.0009i$ | 0.8749, 0.9323 | $0.8451 \pm 0.2084i$ |

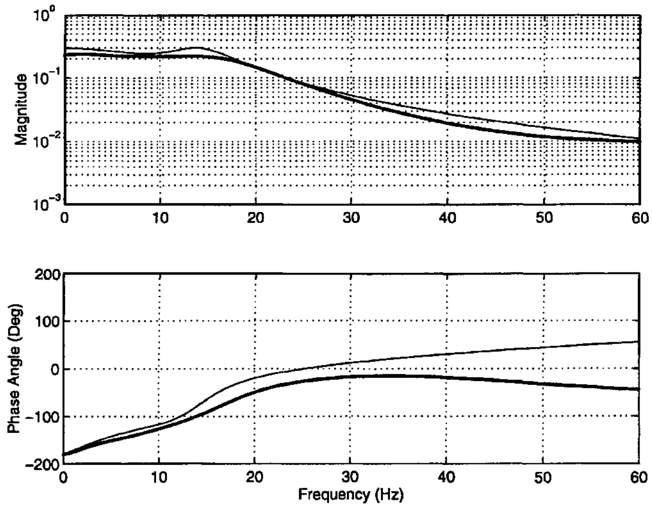


**Fig. 1 Comparison of the closed-loop analytical (thin line) and reconstructed (thick line) input-1/output-1 FRFs. The reconstructed FRF is obtained using the identified system matrices.**

Table 1 compares identified system eigenvalues with true ones from numerical simulations. The results show perfect match when there is no noise and quite good agreement even with 1% of processing and measurement noises (1% noise variance). Figure 1 also shows the comparison of the closed-loop analytical (thin line) and reconstructed (thick line) input-1/output-1 FRFs.

Five experiments also were performed. In each experiment, only one of the five actuators had a single random reference input, and all others had zero reference input. A total of 4096 data points at a sampling rate of 250 Hz from each sensor were recorded. Six FRFs from these single input/six output data can be derived. The experiment is repeated by sending the same single random input to a different actuator each time. Thirty FRFs were obtained. The order of the matrix polynomial was set to 13. The identified eigenvalues from testing are shown in Table 1.

## Concluding Remarks

A method of identifying a linear state-space model of a plant from closed-loop frequency response data with known feedback dynamics is developed. The main contribution is the derivation of the relationship between the open-loop-system Markov parameters and the closed-loop frequency response function for a linear stochastic system. Numerical simulations and experimental results of a highly unstable large-gap magnetic suspension system are presented.

## Appendix: Analytical System and Controller Matrices

The matrices of Eqs. (1), (2), (5), and (6) are as follows:

$$A = \begin{bmatrix} 1.1687 & 0.0006 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 1.1629 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0001 & 1.0178 & -0.0017 & -0.0037 & 0.0021 & 0.0074 & -0.0127 & 0.0112 & 0.0006 \\ -0.0000 & 0.0000 & 0.0001 & 1.0051 & 0.0001 & 0.0295 & 0.0006 & 0.0015 & -0.0011 & 0.0003 \\ 0.0000 & 0.0002 & -0.0004 & 0.0008 & 1.0106 & -0.0018 & 0.0223 & 0.0066 & -0.0039 & 0.0030 \\ 0.0000 & -0.0000 & -0.0021 & -0.0240 & 0.0005 & 0.9908 & 0.0028 & -0.0010 & 0.0003 & -0.0011 \\ 0.0000 & -0.0001 & -0.0064 & -0.0001 & -0.0213 & -0.0041 & 0.9692 & 0.0064 & 0.0004 & 0.0003 \\ -0.0000 & -0.0000 & 0.0109 & -0.0009 & -0.0045 & 0.0021 & 0.0050 & 0.9260 & -0.0549 & 0.0028 \\ 0.0000 & -0.0000 & -0.0086 & 0.0009 & 0.0032 & 0.0009 & 0.0031 & -0.0589 & 0.9125 & -0.0008 \\ 0.0000 & -0.0000 & 0.0004 & 0.0002 & 0.0006 & 0.0012 & 0.0545 & -0.0002 & -0.0002 & 0.8652 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0035 & 0.0706 & 0.0519 & -0.0363 & -0.0633 \\ -0.0434 & -0.0326 & -0.0340 & -0.0425 & -0.0396 \\ 0.0580 & -0.0454 & 0.0983 & -0.0361 & 0.0254 \\ -0.0926 & -0.0315 & 0.0881 & 0.0865 & -0.0218 \\ 0.1160 & 0.0124 & 0.0263 & 0.0982 & -0.0242 \\ -0.1015 & -0.0368 & 0.1033 & 0.0854 & -0.0154 \\ 0.1373 & 0.0057 & 0.0719 & 0.0859 & -0.0066 \\ -0.0159 & -0.0637 & -0.1326 & 0.1165 & 0.0625 \\ 0.0158 & -0.1531 & -0.0261 & 0.0041 & 0.1245 \\ -0.0484 & -0.0800 & -0.0513 & -0.0553 & -0.1009 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.0313 & 0.4029 & -0.0469 & 0.2269 & -0.0381 & -0.1961 & 0.1274 & -0.0363 & 0.0198 & -0.1513 \\ 0.0291 & -0.4213 & 0.0006 & 0.2248 & 0.0290 & -0.2097 & -0.1079 & -0.0130 & 0.0297 & 0.1502 \\ -0.4423 & 0.1071 & 0.1809 & 0.0553 & 0.0669 & -0.0618 & -0.0906 & -0.0418 & -0.2228 & -0.0472 \\ -0.4254 & -0.1184 & -0.1787 & -0.0092 & -0.0829 & 0.0200 & 0.1217 & -0.2197 & -0.0559 & 0.0630 \\ 0.4495 & -0.0763 & 0.0574 & 0.0273 & -0.1861 & -0.0400 & 0.1239 & 0.2109 & 0.0827 & 0.0464 \\ 0.3889 & 0.1015 & -0.0614 & 0.0085 & 0.1739 & 0.0012 & -0.1277 & 0.0386 & 0.1913 & -0.0634 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0.3333 & 0 & 0 & 0 & 0 \\ 0 & 0.3333 & 0 & 0 & 0 \\ 0 & 0 & 0.6000 & 0 & 0 \\ 0 & 0 & 0 & 0.6000 & 0 \\ 0 & 0 & 0 & 0 & 0.6000 \end{bmatrix} \quad B_d = \begin{bmatrix} -0.0206 & 0.0206 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0098 & -0.0098 & 0.0098 & 0.0098 \\ 0 & 0 & 0.0003 & -0.0003 & -0.0003 & 0.0003 \\ 0 & 0 & -0.0003 & 0.0003 & -0.0003 & 0.0003 \\ 0.0004 & 0.0004 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_d = 1.0e+03^* \begin{bmatrix} 0.0796 & 0.0000 & 7.3872 & 0.0000 & -5.5493 \\ 0.1032 & 0.0716 & -5.9772 & 4.3222 & -1.7160 \\ 0.0886 & 0.0442 & 2.2836 & -6.9917 & 4.4907 \\ 0.0886 & -0.0442 & 2.2836 & 6.9917 & 4.4907 \\ 0.1032 & -0.0716 & -5.9772 & -4.3222 & -1.7160 \end{bmatrix}$$

$$D_d = \begin{bmatrix} 10.8171 & 3.9903 & -7.0133 & 7.0133 & 7.0133 & -7.0133 \\ 6.7151 & -2.1362 & 11.2687 & -8.3349 & -3.0144 & 0.0807 \\ -2.1923 & -9.7904 & -7.9381 & 9.7505 & -5.4144 & 3.6020 \\ -2.1923 & -9.7904 & 3.6020 & -5.4144 & 9.7505 & -7.9381 \\ 6.7151 & -2.1362 & 0.0807 & -3.0144 & -8.3349 & 11.2687 \end{bmatrix}$$


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## References


[1] Juang, J.-N., *Applied System Identification*, PTR Prentice–Hall, Englewood Cliffs, NJ, 1994, pp. 229–252.

[2] Huang, J.-K., Hsiao, M.-H., and Cox, D. E., "Indirect Identification of Linear Stochastic Systems with Known Feedback Dynamics," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 4, pp. 836–841.